A GENERALIZED CONTINUOUS MODEL FOR RANDOM MARKETS

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A generalized continuous economic model is proposed for random markets. In this model, agents interact by pairs and exchange their money in a random way. A parameter controls the effectiveness of the transactions between the agents. We show in a rigorous way that this type of markets reach their asymptotic equilibrium on the exponential wealth distribution.

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1. Introduction

In the last years, different techniques and models from statistical physics are being successfully applied to explain some real data observed in economy [1]. Concretely, it has been reported that western (capitalistic) economies can be divided into two distinct groups according to the incomes of people [2]. The 95% of the population, the middle and lower economic classes of society, arranges their incomes in an exponential wealth distribution. The incomes of the rest of the population, the 5% of individuals, fit a power law distribution.

Different models have been used to explain the origin of these wealth distributions. From a macroscopic point of view, it is necessary to take into account that markets have an intrinsic random ingredient as a consequence of the interaction of an undetermined ensemble of agents which are performing an undetermined number of commercial transactions at each moment. A kind of models considering this unknowledge associated to markets are the gas-like models [3]. These random models interpret economic exchanges of money between agents similarly to collisions in a gas where particles share their energy. In order to explain the two different types of statistical behavior before mentioned, different gas-like models have been proposed. On the one hand, the exponential distribution can be obtained by supposing a gas of agents that trade with money in binary collisions, or in first-neighbor interactions, and where the agents are selected in a random, deterministic or chaotic way [4,5,6]. On the other hand, the power law behavior can be simulated by introducing inhomogeneity in the gas by different methods, through the breaking of the pairing symmetry in the exchange rules [6] or through the saving propensity of the agents [7].

In this work, we consider a continuous version of one of such homogeneous gas-like models [8] which we generalize to a situation where the agents present a control parameter to decide the degree of interaction with the rest of economic agents. A zero value of the parameter will represent a null interaction of each agent with the rest of the ensemble and a value equal to unity will mean the total disponibility of each agent for the interaction with the whole gas. Under these circumstances, we are interested in explaining the appearance of the exponential (Gibbs) distribution for all the intermediary values, between zero and one, of the parameter value. This situation can represent the ubiquity of this distribution in many natural phenomena but in particular in the random markets. Thus, the rigorous analytical results here exposed will help us to enlighten the behavior of real data for income distributions but also to understand the computational statistical behavior found in such type of economic models which decay to the exponential wealth distribution in their asymptotic regime independently of the initial wealth distribution given to the system.

2. The continuous gas-like model. Properties

We consider an ensemble of economic agents trading with money in a random manner [4]. This is one of the simplest gas-like models, in which an initial amount of money is given to each agent, let us suppose the same to each one. Then, pairs of agents are randomly chosen and they exchange their money also in a random way. When the gas evolves under these conditions, the exponential distribution appears as the asymptotic wealth distribution. In this model, the microdynamics is conservative because the local interactions conserve the money. Hence, the macrodynamics is also conservative and the total amount of money is constant in time.

The discrete version of this model is as follows [4]. The trading rules for each

interacting pair (m_i, m_j) of the ensemble of N economic agents can be written as

$$\begin{aligned} m_i' &= \epsilon\,(m_i + m_j), \\ m_j' &= (1-\epsilon)(m_i + m_j), \\ i, j &= 1 \ldots N, \end{aligned} \tag{2.1}$$

where ϵ is a random number in the interval $(0, 1)$. The agents (i, j) are randomly chosen. Their initial money (m_i, m_j), at time t, is transformed after the interaction in (m_i', m_j') at time $t+1$. The asymptotic distribution $p_f(m)$, obtained by numerical simulations, is the exponential (Boltzmann-Gibbs) distribution,

$$p_f(m) = \beta \exp(-\beta\, m), \qquad \text{with} \qquad \beta = 1/ < m >_{gas}, \tag{2.2}$$

where $p_f(m)\mathrm{d}m$ denotes the PDF (*probability density function*), i.e. the probability of finding an agent with money (or energy in a gas system) between m and $m+\mathrm{d}m$. Evidently, this PDF is normalized, $||p_f|| = \int_0^\infty p_f(m)\mathrm{d}m = 1$. The mean value of the wealth, $< m >_{gas}$, can be easily calculated directly from the gas by $< m >_{gas} = \sum_i m_i/N$.

The continuous version of this model [8] considers the evolution of an initial wealth distribution $p_0(m)$ at each time step n under the action of an operator T. Thus, the system evolves from time n to time $n+1$ to asymptotically reach the equilibrium distribution $p_f(m)$, i.e.

$$\lim_{n\to\infty} T^n\,(p_0(m)) \to p_f(m). \tag{2.3}$$

In this particular case, $p_f(m)$ is the exponential distribution with the same average value $< p_f >$ than the initial one $< p_0 >$, due to the local and total richness conservation.

The derivation of the operator T is as follows [8]. Suppose that p_n is the wealth distribution in the ensemble at time n. The probability to have a quantity of money x at time $n+1$ will be the sum of the probabilities of all those pairs of agents (u, v) able to produce the quantity x after their interaction, that is, all the pairs verifying $u+v > x$. Thus, the probability that two of these agents with money (u, v) interact between them is $p_n(u) * p_n(v)$. Their exchange is totally random and then they can give rise with equal probability to any value x comprised in the interval $(0, u+v)$. Therefore, the probability to obtain a particular x (with $x < u+v$) for the interacting pair (u, v) will be $p_n(u) * p_n(v)/(u+v)$. Then, T has the form of a nonlinear integral operator,

$$p_{n+1}(x) = Tp_n(x) = \iint_{u+v>x} \frac{p_n(u)p_n(v)}{u+v}\,\mathrm{d}u\mathrm{d}v\,. \tag{2.4}$$

If we suppose T acting in the PDFs space, it has been proved [9] that T conserves the mean wealth of the system, $< Tp > = < p >$. It also conserves the norm ($||\cdot||$), i.e. T maintains the total number of agents of the system, $||Tp|| = ||p|| = 1$, that by extension implies the conservation of the total richness of the system. We have also shown that the exponential distribution $p_f(x)$ with the right average value is the

only steady state of T, i.e. $Tp_f = p_f$. Computations also seem to suggest that other high period orbits do not exist. In consequence, it can be argued that the relation (2.3) is true. We proceed to recall these properties and to establish some new ones.

2.1. *Properties of the operator T. Recall*

First, in order to set up the adequate mathematical framework, we provide the following definitions.

Definition 2.1. We introduce the space L_1^+ of positive functions (wealth distributions) in the interval $[0, \infty)$,

$$L_1^+[0, \infty) = \{y : [0, \infty) \to R^+ \cup \{0\}, \ \ ||y|| < \infty\},$$

with norm

$$||y|| = \int_0^\infty y(x)dx.$$

Definition 2.2. We define the mean richness $< x >_y$ associated to a wealth distribution $y \in L_1^+[0, \infty)$ as the mean value of x for the distribution y. In the rest of the paper, we will represent it by $< y >$. Then,

$$< y > \equiv < x >_y = ||xy(x)|| = \int_0^\infty xy(x)dx.$$

Definition 2.3. For $x \geq 0$ and $y \in L_1^+[0, \infty)$ the action of operator T on y is defined by

$$T(y(x)) = \int\int_{S(x)} dudv \frac{y(u)y(v)}{u+v},$$

with $S(x)$ the region of the plane representing the pairs of agents (u, v) which can generate a richness x after their trading, i.e.

$$S(x) = \{(u, v), \ \ u, v > 0, \ \ u + v > x\}.$$

Now, we remind the following results recently presented in Ref. [9].

Theorem 2.1. *For any $y \in L_1^+[0, \infty)$ we have that $||Ty|| = ||y||^2$. (It means that the number of agents in the economic system is conserved in time). In particular, consider the subset of PDFs in $L_1^+[0, \infty)$, i.e. the unit sphere $B = \{y \in L_1^+[0, \infty)$, $||y|| = 1\}$. Observe that if $y \in B$ then $Ty \in B$.*

Theorem 2.2. *The mean value $< y >$ of a PDF y is conserved, that is $< Ty >=< y >$ for any $y \in B$. (It means that the mean wealth, and by extension the total richness, of the economic system are preserved in time).*

Theorem 2.3. *Apart from $y = 0$, the one-parameter family of functions $y_\alpha(x) = \alpha e^{-\alpha x}$, $\alpha > 0$, are the unique fixed points of T in the space $L_1^+[0, \infty)$.*

Proposition 2.1. *The operator T is Lipschitz continuous in B with Lipschitz constant ≤ 2. It means that if we take $y, w \in B$, then*

$$||Ty - Tw|| \leq 2\,||y - w||.$$

Proposition 2.2. *For any $y \in L_1^+[0,\infty)$ and $m \leq n \in N$, it holds that $(-1)^m (T^n y)^{(m)} \in L_1^+[0,\infty)$. It implies that if $T^n y = y$ then y is a completely monotonic function.*

2.2. *Other properties of T*

Evidently, our main interest resides in the action of the operator T in the subset of PDFs. We give here other properties for this restriction of T in B, although most of these properties are clearly valid for the whole space $L_1^+[0,\infty)$.

Proposition 2.3. *Suppose that $y \in B$ then Ty is a decreasing function.*

Proof. Assume that $x_1, x_2 \geq 0$ and $x_1 \leq x_2$ then $\{(u,v),\ \ u,v>0,\ \ u+v>x_2\} \subset \{(u,v),\ \ u,v>0,\ \ u+v>x_1\}$ and so

$$Ty(x_1) = \iint_{u+v>x_1} \frac{y(u)y(v)}{u+v}\,\mathrm{d}u\mathrm{d}v \geq \iint_{u+v>x_2} \frac{y(u)y(v)}{u+v}\,\mathrm{d}u\mathrm{d}v = Ty(x_2)\,.$$

Therefore Ty for every function $y \in B$ is a decreasing function. This is a particular case of Proposition 2.2 for $n = m = 1$. □

Corollary 2.1. *Suppose that $y \in B$ is a not decreasing function then $\forall n \in \mathbb{N}$: $T^n y \neq y$.*

Proof. It follows directly from Proposition 2.3. □

Proposition 2.4.

1. *For every $y, w \in B$ we have $||Ty - Tw|| \leq 2$.*
2. *For some members $y, w \in B$, $||Ty - Tw|| = ||y - w||$, hence T is not a contraction.*

Proof.

1. It is clear that $||Ty - Tw|| \leq ||Ty|| + ||Tw|| = 1 + 1 = 2$.
2. Take $y = \alpha e^{-\alpha x}$ and $w = \beta e^{-\beta x}$, with $\alpha, \beta > 0$, which belong to B. In this case, we can easily see that $||Ty - Tw|| = ||\alpha e^{-\alpha x} - \beta e^{-\beta x}|| = ||y - w||$. □

Example 2.1. Take $y(x) = \frac{1}{(1+x)^2}$ and $w(x) = e^{-x}$ which belong to B. By using Mathematica, it is seen that $||y - w|| = 0.407264$ and $||Ty - Tw|| = 0.505669$. Then, $||Ty - Tw|| > ||y - w||$ which confirms again that T is not a contraction in B.

Conjecture: For any $y \in B$, we guess by simulation of many examples the truth of the following relation:

$$\lim_{n\to\infty} T^n y(x) = \begin{cases} \delta e^{-\delta x} & \textit{with} \quad \delta = 1/ < y > , \\ & \textit{or} \\ 0^+ & \textit{when} \quad < y >= +\infty \, . \end{cases}$$

Let us observe that the above pointwise limit of $T^n y$ when $n \to \infty$ can be outside of B in the case that $< y >= +\infty$. See the next example.

Example 2.2. Take $y(x) = \frac{1}{(1+x)^2}$ which belongs to B, with $< y >= +\infty$. Evidently, $T^n y \in B$ for all n. But it can be graphically guessed that $\lim_{n\to\infty} T^n y(x) = 0^+ \notin B$. See Figure 1.

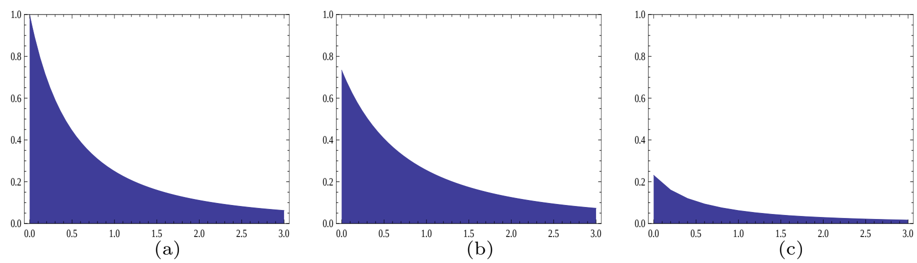


Fig. 1. (a) $y(x) = \frac{1}{(1+x)^2}$, (b) $Ty(x)$, (c) $T^2 y(x)$.

Theorem 2.4. *Suppose that* $\lim_{n\to\infty} ||T^n y(x) - \mu(x)|| = 0$*, and* $\mu(x)$ *is a continuous function, then* $\mu(x)$ *is the fixed point of the operator* T *for the initial condition* $y(x) \in B$*. In other words,* $\mu(x) = \delta e^{-\delta x}$ *with* $\delta = 1/ < y >$*.*

Proof. First we prove that for each $\epsilon > 0$ there exists M so that $\forall n > M$: $||T^n y - T\mu|| < \epsilon$. Since $\lim_{n\to\infty} ||T^n y(x) - \mu(x)|| = 0$, we can say that $\exists N$ such that $\forall n > N$: $||T^{n-1} y - \mu|| < \epsilon/2$. Now, we choose $M = N$. Then, due to the Lipschitz continuity of T, we have

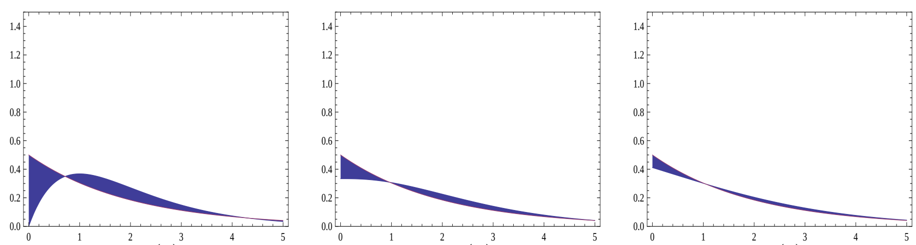
$$\forall n > M : ||T^n y - T\mu|| \leq 2||T^{n-1} y - \mu|| < 2 \cdot \frac{\epsilon}{2} = \epsilon \, .$$

Now, by uniqueness of the limit, it implies that $T\mu = \mu$. Therefore, for the initial condition $y(x) \in B$, $\mu(x)$ is the fixed point of T, and by Theorem 2.3 it means that $\mu(x) = \delta e^{-\delta x}$ with $\delta = 1/ < y >$. $\square$

As particular examples, we present in Figs. 2 and 3 the graphical evidence of the convergence suggested in the former Theorem. Many other cases have been studied with a similar behavior. Remark that in Example 2.2 there is pointwise

convergence to the null function ($\mu = 0^+$) but there is no convergence in norm L_1, in fact $||T^n y - \mu|| = 1$ for all n.

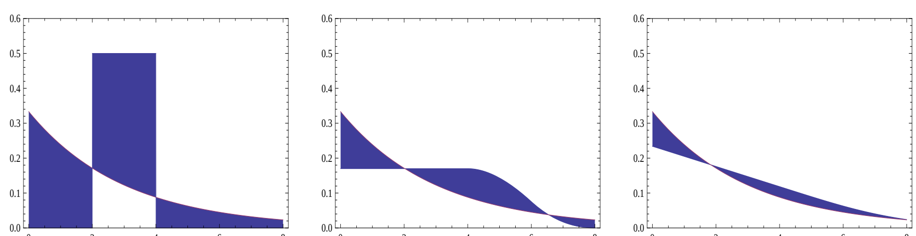
Example 2.3. Take the Gamma distribution $y(x) = xe^{-x}$, so that $y \in B$ and $\delta = \frac{1}{2}$, then in this case $\mu(x) = \frac{1}{2}e^{-\frac{1}{2}x}$. We find numerically that $||y - \mu|| = 0.368226$, $||Ty - \mu|| = 0.185608$, $||T^2 y - \mu|| = 0.103225$, $||T^3 y - \mu|| = 0.061195$, $||T^4 y - \mu|| = 0.037675$, and so on. It is shown in Fig. 2. Then we can guess that $\lim_{n\to\infty} ||T^n y - \mu|| = 0$.



(a) (b) (c)

Fig. 2. Plot of $y(x) = xe^{-x}$, T-iterates of y and $\mu(x) = \frac{1}{2}e^{-\frac{1}{2}x}$. (a) $||y - \mu||$, (b) $||Ty - \mu||$, (c) $||T^2 y - \mu||$.

Example 2.4. Assume now the rectangular distribution: $y(x) = \frac{1}{2}$ if $2 < x < 4$, and $y(x) = 0$ otherwise. So, $y \in B$ and $\delta = \frac{1}{3}$, then in this case $\mu(x) = \frac{1}{3}e^{-\frac{1}{3}x}$. We find numerically that $||y-\mu|| > ||Ty-\mu|| > ||T^2 y-\mu|| > ||T^3 y-\mu||$, and so on. It is shown in Fig. 3. Then we can also guess in this case that $\lim_{n\to\infty} ||T^n y - \mu|| = 0$.



(a) (b) (c)

Fig. 3. Plot of $y(x) = \frac{1}{2}$ if $2 < x < 4$, and $y(x) = 0$ otherwise, T-iterates of y and $\mu(x) = \frac{1}{3}e^{-\frac{1}{3}x}$. (a) $||y - \mu||$, (b) $||Ty - \mu||$, (c) $||T^2 y - \mu||$.

Summarizing, the dynamical evolution of T in $L_1^+[0,\infty)$ is roughly given by the property: $||Ty|| = ||y||^2$, which divides the space $L_1^+[0,\infty)$ in three regions: the interior of the unit ball $B_{in} = \{y \in L_1^+[0,\infty), ||y|| < 1\}$, the unit sphere B, and the

exterior of the unit ball $B_{ex} = \{y \in L_1^+[0,\infty),\ ||y|| > 1\}$. Accordingly, the recursion

$$y_n(x) = T^n y_0(x), \qquad \text{with} \quad y_0(x) \in L_1^+[0,\infty),$$

presents four different behaviors depending on y_0:

- If $y_0 \in B_{in}$, then $\lim_{n\to\infty} ||y_n(x)|| = 0$.
- If $y_0 \in B$ and $< y_0 >$ is finite, then $\lim_{n\to\infty} ||y_n(x) - \delta e^{-\delta x}|| = 0$, with $\delta = \frac{1}{<y_0>}$. Observe that the succession y_n remains in B $\forall n$.
- If $y_0 \in B$ and $< y_0 >$ is infinite, then $\lim_{n\to\infty} ||y_n(x) - 0^+|| = 1$. Observe that in this case the succession y_n remains in B $\forall n$ and the series y_n does not converge in the norm $||\cdot||$ to its pointwise limit $0^+ \notin B$.
- If $y_0 \in B_{ex}$, then $\lim_{n\to\infty} ||y_n(x)|| = \infty$.

Hence, the only fixed points of the system are $y = 0$ and $\delta e^{-\delta x}$. They are asymptotically reached depending on the initial average value $< y_0 >$, which determines the final equilibrium. We proceed now to show that this behavior is essentially maintained in the extension of this model for more general random markets.

3. Generalized continuous model for random markets

Let us observe that the action of operator T can receive a macroscopic interpretation respect to Eqs. (2.1) in the sense that, each iteration of the operator T means that many interactions, of order $N/2$, have taken place between different pairs of agents. As n indicates the time evolution of T, we can roughly assume that $t \approx N * n/2$, where t follows the microscopic evolution of the individual transactions (or collisions) between the agents (this alternative microscopic interpretation can be seen in [10]).

Let us think now that many of the economical transactions planned in markets are not successful and they are finally frustrated. It means that markets are not totally effective. We can reflect this fact in our models in a qualitative way by defining a parameter $\lambda \in [0,1]$ which indicates the *degree of effectiveness* of the random market. When $\lambda = 1$ the market will have total effectiveness and all the operations will be performed under the action of the random rules (2.1). The evolution of the system in this case is given by the operator T. When $\lambda = 0$, all the operations become frustrated, there is no exchange of money between the agents and then the market stays frozen in its original state. The operator representing this type of dynamics is just the identity operator. Therefore, we can establish a *generalized continuous economic model* whose evolution in the PDFs space is determined by the operator T_λ, which depends on the parameter λ as follows:

Definition 3.1.

$$T_\lambda y(x) = (1-\lambda) y(x) + \lambda T y(x), \qquad \textit{with} \qquad \lambda \in [0,1].$$

Observe that the parameter $(1-\lambda)$ can also be interpreted as a kind of *saving propensity* of the agents, in such a way that for $\lambda = 1$ they do not save anything and

they game all their resources, and for $\lambda = 0$ they save the totality of their money and then all the transactions are frustrated and the market stays in a frozen state.

3.1. *Properties of the operator T_λ*

Now, we proceed to present the properties of the operator T_λ, which shows a dynamical behavior essentially similar to the behavior of T. Concretely, one of our main results is that the exponential distribution is also the asymptotic wealth distribution reached by the system governed by T_λ, independently of the effectiveness λ of the random market.

Let us observe that $T_\lambda = I$ for $\lambda = 0$ and $T_\lambda = T$ for $\lambda = 1$, where I is the identity operator.

Proposition 3.1. *T_λ conserves the norm, i.e., for each $y \in B$, we have $T_\lambda y \in B$.*

Proof. Consider $y(x) \geq 0$, then $T_\lambda y(x) \geq 0$. The norm is:

$$||T_\lambda y(x)|| = (1-\lambda)||y(x)|| + \lambda||Ty(x)|| = (1-\lambda) + \lambda = 1. \qquad \square$$

Proposition 3.2. *T_λ conserves the average value of $y \in B$, i.e. $< y > = < T_\lambda y >$, where $< y >$ represents the mean value $< x >_y$ as expressed in Definition 2.2.*

Proof. Take $y \in B$, then

$$< T_\lambda y > = (1-\lambda) < y > + \lambda < Ty > = (1-\lambda) < y > + \lambda < y > = < y > . \qquad \square$$

Proposition 3.3. *The operator T_λ is Lipschitz continuous in B with a Lipschitz factor α such that $\alpha \leq 1 + \lambda$.*

Proof. Suppose that $y, w \in B$, then

$$||T_\lambda y - T_\lambda w|| = ||(1-\lambda)y + \lambda Ty - (1-\lambda)w - \lambda Tw|| \leq$$

$$\leq (1-\lambda)||y - w|| + \lambda||Ty - Tw|| \leq$$

$$\leq (1-\lambda)||y - w|| + 2\lambda||y - w|| =$$

$$\leq (1+\lambda)||y - w||. \qquad \square$$

Theorem 3.1. *For any $\lambda \in (0, 1)$, the operators T and T_λ have the same fixed points.*

Proof. (i) Suppose y is a fixed point of T, i.e. $Ty = y$. Then we have $T_\lambda y = y$, because $T_\lambda y = (1-\lambda)y + \lambda Ty = y - \lambda y + \lambda y = y$.
(ii) Now suppose y is a fixed point of T_λ, i.e. $T_\lambda y = y$. Then we have $Ty = y$, because $\lambda Ty = T_\lambda y - (1-\lambda)y = y - y + \lambda y = \lambda y$. $\square$

Corollary 3.1. *The function $y(x) = 0$ and the family of exponential distributions $y_\delta(x) = \delta e^{-\delta x}$, $\delta > 0$, are the only fixed points of T_λ in $L_1^+[0, \infty)$, with $\lambda \in (0, 1]$.*

Proof. It follows from Theorems 3.1 and 2.3. □

Theorem 3.2. *Suppose that for a given $\lambda \in (0, 1)$ we have $\lim_{n\to\infty} ||T_\lambda^n y(x) - \mu(x)|| = 0$, with $\mu(x)$ a continuous function, then $\mu(x)$ should be the fixed point of the operator T_λ for the initial condition $y(x) \in B$. In other words, $\mu(x) = \delta e^{-\delta x}$ with $\delta = \frac{1}{<y>}$.*

Proof. Identical to the proof of Theorem 2.4 by changing T by T_λ and taking into account Theorem 3.1. □

Example 3.1. Take again the Gamma distribution $y(x) = xe^{-x}$, so that $y \in B$ and $\delta = \frac{1}{2}$, then in this case $\mu(x) = \frac{1}{2}e^{-\frac{1}{2}x}$. For $\lambda = 0.5$, we find numerically that $||y - \mu|| = 0.368226$, $||T_\lambda y - \mu|| = 0.273011$, $||T_\lambda^2 y - \mu|| = 0.206554$, $||T_\lambda^3 y - \mu|| = 0.158701$, and so on. It is shown in Fig. 4. Then we can guess that $\lim_{n\to\infty} ||T_\lambda^n y - \mu|| = 0$. Observe that in this case $T_\lambda y$ can be a non-decreasing function.

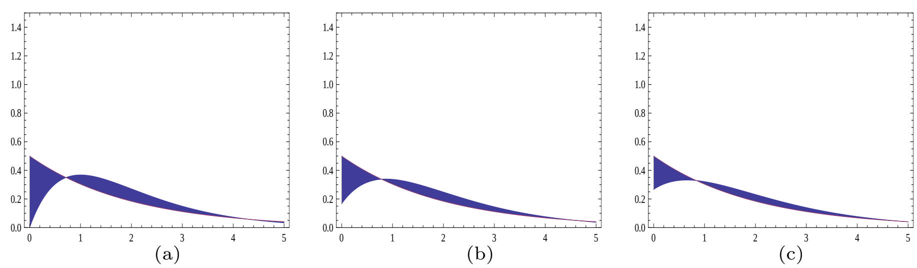


Fig. 4. Plot of $y(x) = xe^{-x}$, T_λ-iterates of y for $\lambda = 0.5$ and $\mu(x) = \frac{1}{2}e^{-\frac{1}{2}x}$. (a) $||y - \mu||$, (b) $||T_\lambda y - \mu||$, (c) $||T_\lambda^2 y - \mu||$.

4. Conclusions

In this work, a continuous economic model recently introduced [8] has been generalized. This model takes into account idealistic characteristics of the markets, where agents interact by pairs and exchange their money in a random way. Also, the model implements a parameter that gives an idea of the effectiveness of the agents when trading between them. A perfect effectiveness means the total availability of the agents wealth to be gamed in their transactions and a null effectiveness gives rise to a frozen market where each agent keeps intact his money. We have shown in a rigorous manner that it does not matter the degree of effectiveness of the market for the final statistical result. In all the cases, these random markets evolve toward their asymptotic equilibrium, that is, the exponential wealth distribution.

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